AN AXIOMATIC APPROACH FOR DEGENERATIONS IN TRIANGULATED CATEGORIES

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ABSTRACT. We generalise Yoshino's definition of a degeneration of two Cohen Macaulay modules to a definition of degeneration between two objects in a triangulated category. We derive some natural properties for the triangulated category and the degeneration under which the Yoshino-style degeneration is equivalent to the degeneration defined by a specific distinguished triangle analogous to Zwara's characterisation of degeneration in module varieties.

INTRODUCTION

For an algebra A over a field k, A-modules of fixed dimension d over k are k-vector spaces together with an action of A. Fix k^d and then the conditions reflecting these properties can be formulated in terms of vanishing of a finite number of polynomial equations. Hence, the points of the algebraic variety defined by these polynomial equations correspond to the possible A-module structures on k^d. Two such structures are isomorphic if and only if they belong to the same $Gl_d(k)$-orbit. The variety together with the action of the linear group is called the module variety $mod_d(A)$. If M corresponds to the point m and if N corresponds to the point n, then we say that M degenerates to N if n is in the Zariski-closure of the $Gl_d(k)$-orbit of m, and we denote in this case $M \leq_{deg} N$.

Riedtmann and Zwara showed in [12, 19] that $M \leq_{deg} N$ if and only if there is an A-module Z and an exact sequence

$$0 \longrightarrow Z \longrightarrow Z \oplus M \longrightarrow N \longrightarrow 0$$

and this is equivalent to the existence of an A-module Z' and an exact sequence

$$0 \longrightarrow N \longrightarrow Z' \oplus M \longrightarrow Z' \longrightarrow 0.$$

The concept of degeneration of modules is highly successful in representation theory of algebras.

Various attempts were undertaken to prove an analogous result for triangulated categories. In [3] the first author defined degenerations of complexes of A-modules, and a slight modification yields to a degeneration concept of the derived category. In [5] Jensen, Su and the second author defined a topological space, whose points correspond to objects in $D^-(A)$, and on which a topological group acts. In a parallel development Jensen, Madsen and Su give in [4] another algebraic variety with a group action parameterising objects in the derived category of A_∞-modules over A_∞-algebras. Degeneration can be defined analogously to the module variety situation and then it can be shown in the last two settings that an object M degenerates to an object N if and only if there is an object Z and a distinguished triangle

$$N \longrightarrow Z \oplus M \longrightarrow Z \longrightarrow N[1].$$

This second condition on the existence of distinguished triangles can be used to define some relation $\leq_\Delta$ on isomorphism classes of objects in triangulated categories $\mathcal{T}$. It is shown

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in [6] that under certain conditions the relation $\leq_\Delta$ is a partial order on the isomorphism classes of objects of $\mathcal{T}$.

As for stable categories Yoshino gave a definition of stable degeneration $\leq_{stdeg}$ in [16] building on earlier work on an alternative approach for module varieties (cf [14, 15]). For stable categories of maximal Cohen-Macaulay modules over isolated singularities Yoshino proved that $\leq_{stdeg}$ is equivalent to the partial order $\leq_\Delta$. Yoshino's construction uses that an object M degenerates to N if and only if there is a line through N so that the generic point of the line is isomorphic to the generic point generated by M. Modelling this gives a criterion on the existence of a special object over an extension of the base category.

A geometrically inspired notion of degeneration was still missing. The purpose of this note is to develop such a notion which works in a very general triangulated setting, and our main result Theorem 1 shows that this geometrically inspired notion of degeneration is equivalent to $\leq_\Delta$. As a by-product the notion also works for bounded derived categories improving in this way [3], [5] and [4]. We proceed analogously to Yoshino's approach and formalise the existence of a line to a functor to an extension category together with an element in its centre, and the condition of the generic point to a condition on the localisation category. The main definition is Definition 3 and the definition of a degeneration is given in Definition 4.

We give an outline of the structure of the paper. In Section 1 the definition of triangulated degeneration is given. In Section 2 we give some technical details valid for triangulated categories which we use in the sequel and which we recall for the reader's convenience. Our main result is then formulated and proved in Section 3, which contains Theorem 1 and its proof fills the entire section. We conclude with Section 4 which contains some consequences and remarks on degeneration for triangulated categories.

1. Defining degeneration

1.1. **The ring theoretic degeneration axioms.** We shall define a system of axioms modelling the classical situation. First recall Yoshino's definition of module degeneration.

Definition 1. (Yoshino [15]) Let k be a field and let A be a k-algebra. Then for all finitely generated A-modules M and N we say that M degenerates to N along a discrete valuation ring if there is a discrete valuation ring (V, tV, k) that is a k-algebra (where t is a prime element) and a finitely generated $A \otimes_k V$-module Q such that

(1) Q is flat as a V-module
(2) $Q/tQ \simeq N$ as an A-module
(3) $Q[\frac{1}{t}] \simeq M \otimes_k V[\frac{1}{t}]$ as an $A \otimes V[\frac{1}{t}]$-module.

Yoshino shows in [15] that M degenerates to N along a discrete valuation ring if and only if there is a short exact sequence of finitely generated A-modules

$$0 \to Z \xrightarrow{\begin{pmatrix} v \\ u \end{pmatrix}} Z \oplus M \longrightarrow N \to 0$$

such that v is a nilpotent endomorphism of Z.

Yoshino generalizes in [16] this concept in the obvious way to define degeneration in the (triangulated !) stable category of maximal Cohen-Macaulay modules over a commutative local Gorenstein k-algebra.

We transpose this to general triangulated categories in a categorical framework. Observe first the following remark.

Remark 2. Let $\mathcal{C}$ be a triangulated category and let t be an element of its center, i.e. a natural transformation of triangulated functors $t : \mathrm{id}_{\mathcal{C}} \longrightarrow \mathrm{id}_{\mathcal{C}}$, where $\mathrm{id}_{\mathcal{C}}$ is the identity functor on $\mathcal{C}$. Then $\Sigma := \{t_X^n |\ X \text{ object of } \mathcal{C},\ n \in \mathbb{N}\}$ is, in the terminology of Verdier (see [13, Section II.2]), a multiplicative system of $\mathcal{C}$ compatible with the triangulation. The localization of $\mathcal{C}$ with respect to Σ, in the sense of Gabriel-Zisman [2], is denoted by

$\mathcal{C}[t^{-1}]$ in the sequel and is then a triangulated category such that the canonical functor $p : \mathcal{C} \longrightarrow \mathcal{C}[t^{-1}]$ is triangulated. Note that the Hom spaces in $\mathcal{C}[t^{-1}]$ are sets, so that this is a proper category. If $\mathcal{D}$ is any triangulated category, then giving a triangulated functor $\Psi : \mathcal{C}[t^{-1}] \longrightarrow \mathcal{D}$ is equivalent to giving a triangulated functor, which we denote the same, $\Psi : \mathcal{C} \longrightarrow \mathcal{D}$ such that $\Psi(t_X)$ is an isomorphism.

Definition 3. Let k be a commutative ring and let $\mathcal{C}_k^\circ$ be a k-linear triangulated category with split idempotents.

A degeneration data for $\mathcal{C}_k^\circ$ is given by

- a triangulated category $\mathcal{C}_k$ with split idempotents and a fully faithful embedding $\mathcal{C}_k^\circ \longrightarrow \mathcal{C}_k$,
- a triangulated category $\mathcal{C}_V$ with split idempotents and a full triangulated subcategory $\mathcal{C}_V^\circ$,
- triangulated functors $\uparrow_k^V : \mathcal{C}_k \longrightarrow \mathcal{C}_V$ and $\Phi : \mathcal{C}_V^\circ \to \mathcal{C}_k$, so that $(\mathcal{C}_k^\circ) \uparrow_k^V \subseteq \mathcal{C}_V^\circ$, when we view $\mathcal{C}_k^\circ$ as a full subcategory of $\mathcal{C}_k$,
- a natural transformation $\mathrm{id}_{\mathcal{C}_V} \xrightarrow{t} \mathrm{id}_{\mathcal{C}_V}$ of triangulated functors

These triangulated categories and functors should satisfy the following axioms:

(1) For each object M of $\mathcal{C}_k^\circ$ the morphism $\Phi(M \uparrow_k^V) \overset{\Phi(t_{M\uparrow_k^V})}{\longrightarrow} \Phi(M \uparrow_k^V)$ is a split monomorphism in $\mathcal{C}_k$.

(2) For all objects M of $\mathcal{C}_k^\circ$ we get $\Phi(\mathrm{cone}(t_{M\uparrow_k^V})) \simeq M$.

All throughout the paper, whenever we have a degeneration data for $\mathcal{C}_k^\circ$ as above, we will see $\mathcal{C}_k^\circ$ as a full subcategory of $\mathcal{C}_k$.

Definition 4. Given two objects M and N of $\mathcal{C}_k^\circ$ we say that M degenerates to N in the categorical sense if there is a degeneration data for $\mathcal{C}_k^\circ$ and an object Q of $\mathcal{C}_V^\circ$ so that

$$p(Q) \simeq p(M \uparrow_k^V) \text{ in } \mathcal{C}_V^\circ[t^{-1}] \text{ and } \Phi(\mathrm{cone}(t_Q)) \simeq N,$$

where $p : \mathcal{C}_V^\circ \longrightarrow \mathcal{C}_V^\circ[t^{-1}]$ is the canonical functor. In this case we write $M \leq_{cdeg} N$.

2. Some tools in triangulated categories

We will use a well-known result in triangulated categories, which can be found in e.g. [11] or [17, Lemma 3.4.5].

Lemma 5. *Let $\mathcal{T}$ be a triangulated category and let*

$$\begin{array}{ccc} A & \overset{\alpha}{\to} & B \\ \downarrow\gamma & & \downarrow\beta \\ C & \overset{\delta}{\to} & D \end{array}$$

be a commutative diagram in $\mathcal{T}$. Then we may complete the horizontal and vertical maps to distinguished triangles and there are morphisms ϵ and φ so that in the diagram

$$\begin{array}{ccccccc} A & \overset{\alpha}{\to} & B & \to & C(\alpha) & \to & A[1] \\ \downarrow\gamma & & \downarrow\beta & & \downarrow\epsilon & & \downarrow\gamma[1] \\ C & \overset{\delta}{\to} & D & \to & C(\delta) & \to & C[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ C(\gamma) & \to & C(\beta) & \to & C(\epsilon) & \to & C(\gamma)[1] \\ \downarrow & & \downarrow & & \downarrow & \text{“}(-)\text{”} & \downarrow \\ A[1] & \overset{\alpha[1]}{\to} & B[1] & \to & C(\alpha)[1] & \to & A[2] \end{array}$$

all horizontal and vertical sequences are distinguished triangles. All squares are commutative, except the lower right one, which is anticommutative.

Idea of the proof. The construction of ϵ is the following. Take the composition $\xi := \beta \circ \alpha = \delta \circ \gamma$ and consider the distinguished triangle

$$A \xrightarrow{\xi} D \to V \to A[1].$$

Then the octahedral axiom applied to the two factorisations of ξ give distinguished triangles

$$C(\alpha) \xrightarrow{\rho} V \longrightarrow C(\beta) \to C(\alpha)[1]$$

and

$$C(\gamma) \to V \xrightarrow{\sigma} C(\delta) \to C(\gamma)[1].$$

Define $\epsilon := \sigma \circ \rho$.

Remark 6. Recall the octahedral axiom. Given three objects X_1, X_2, X_3 and suppose $\alpha_2 : X_1 \longrightarrow X_3$ factorises $\alpha_2 = \alpha_1 \circ \alpha_3$ for $\alpha_3 \in \mathrm{Mor}_{\mathcal{T}}(X_1, X_2)$ and $\alpha_1 \in \mathrm{Mor}_{\mathcal{T}}(X_2, X_3)$. Then forming the triangles

$$X_2 \xrightarrow{\alpha_1} X_3 \xrightarrow{\beta_1} Z_1 \xrightarrow{\gamma_1} X_2[1]$$
$$X_1 \xrightarrow{\alpha_3} X_2 \xrightarrow{\beta_3} Z_3 \xrightarrow{\gamma_3} X_1[1]$$
$$X_1 \xrightarrow{\alpha_2} X_3 \xrightarrow{\beta_2} Z_2 \xrightarrow{\gamma_2} X_1[1]$$

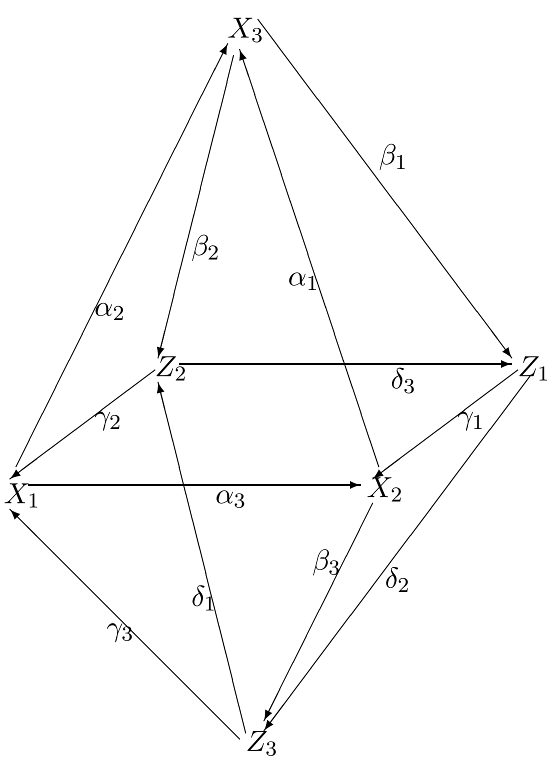
above α_1, α_2 and α_3, there are morphisms $\delta_1 : Z_3 \to Z_2$, $\delta_3 : Z_2 \to Z_1$ and $\delta_2 : Z_1 \to Z_3[1]$ so that

$$Z_3 \xrightarrow{\delta_1} Z_2 \xrightarrow{\delta_3} Z_1 \xrightarrow{\delta_2} Z_3[1]$$

is a distinguished triangle, and so that

$$\gamma_2 \circ \delta_1 = \gamma_3 \ ; \ \delta_3 \circ \beta_2 = \beta_1 \ ; \ \delta_2 = \beta_3[1] \circ \gamma_1 \ ;$$
$$\beta_2 \circ \alpha_1 = \delta_1 \circ \beta_3 \text{ and } \gamma_1 \circ \delta_3 = \alpha_3[1] \circ \gamma_2$$

as illustrated in the following picture.



We remind the reader that the morphisms denoted by γ are all of degree 1, and that also δ_2 is of degree 1.

Recall that the octahedral axiom just guarantees the existence of the distinguished triangle as claimed. No uniqueness is claimed. However, if $\gamma_3 = 0$ then β_3 is a split epimorphism, so that there is a morphism $Z_3 \xrightarrow{\omega_3} X_2$ with $\beta_3 \circ \omega_3 = \mathrm{id}_{Z_3}$. Since δ_1 satisfies $\beta_2 \circ \alpha_1 = \delta_1 \circ \beta_3$, and since split epimorphisms are epimorphisms, this equation determines δ_1. More precisely, multiply with ω_3 from the right to obtain $\beta_2 \circ \alpha_1 \circ \omega_3 = \delta_1 \circ \beta_3 \circ \omega_3 = \delta_1$.

We shall need a technical and well-known lemma on Gabriel-Zisman localisations. For the convenience of the reader we include the short proof.

Lemma 7. *Let X be an object of $\mathcal{C}_V^\circ$. We denote by $p : \mathcal{C}_V^\circ \longrightarrow \mathcal{C}_V^\circ[t^{-1}]$ the natural functor for Gabriel-Zisman localisation. Then the following assertions are equivalent.*

- $p(X) = 0$
- *there is $n \geq 0$ so that $t_X^n = 0$.*

Proof. Put again $\Sigma := \{t_X^n | \ X \text{ object of } \mathcal{C}_1,\ n \in \mathbb{N}\}$. A morphism f of $\mathcal{C}_V^\circ$ has the property $p(f) = 0$ if and only if there is an $s \in \Sigma$ so that $f \circ s = 0$. Hence

$$\begin{aligned} p(X) = 0 \quad &\Leftrightarrow \quad p(\mathrm{id}_X) = 0 \\ &\Leftrightarrow \quad \exists s \in \Sigma : \mathrm{id}_X \circ s = 0 \\ &\Leftrightarrow \quad \text{the zero endomorphism on } X \text{ is in } \Sigma. \end{aligned}$$

Hence,

$$p(X) = 0 \Leftrightarrow \exists n \in \mathbb{N} : t_X^n = 0.$$

□

3. The theorem and its proof

3.1. Categorical degeneration implies triangle degeneration.

Proposition 8. *Let $\mathcal{C}_k^\circ$ be a triangulated k-category with split idempotents and let M and N be two objects of $\mathcal{C}_k^\circ$. Then $M \leq_{cdeg} N$ implies that there is a distinguished triangle*

$$Z \xrightarrow{\begin{pmatrix} v \\ u \end{pmatrix}} Z \oplus M \longrightarrow N \longrightarrow Z[1]$$

in $\mathcal{C}_k^\circ$ with nilpotent endomorphism v of Z.

Proof. Suppose that $M \leq_{cdeg} N$, and there is hence a degeneration data $(\mathcal{C}_k, \mathcal{C}_V, \uparrow_k^V, t)$, an object Q of $\mathcal{C}_V^\circ$, so that $p(Q) \simeq p(M \uparrow_k^V)$ in $\mathcal{C}_V^\circ[t^{-1}]$. Therefore there is a morphism $M \uparrow_k^V \xrightarrow{f} Q$ and a morphism $Q \xrightarrow{g} M \uparrow_k^V$ in $\mathcal{C}_V^\circ$ such that $g \circ f \circ t^r_{M\uparrow_k^V} = t^s_{M\uparrow_k^V}$ and $f \circ g \circ t_Q^m = t_Q^n$, for some $r, s, m, n \in \mathbb{N}$. From the fact that t is a natural transformation, we also get that $t^r_{M\uparrow_k^V} \circ g \circ f = t^s_{M\uparrow_k^V}$ and, by applying the functor $\Phi : \mathcal{C}_V^\circ \longrightarrow \mathcal{C}_k$, we get that $\Phi(f)$ is a split monomorphism.

By completing f to a triangle in $\mathcal{C}_V$

$$M \uparrow_k^V \xrightarrow{f} Q \xrightarrow{\varphi} \mathrm{cone}(f) \longrightarrow M \uparrow_k^V [1],$$

we then get a split distinguished triangle in $\mathcal{C}_k$

$$\Phi(M \uparrow_k^V) \xrightarrow{\Phi(f)} \Phi(Q) \xrightarrow{\Phi(\varphi)} \Phi(\mathrm{cone}(f)) \xrightarrow{0} \Phi(M \uparrow_k^V)[1].$$

By axiom (1) we get a distinguished triangle

$$\Phi(M \uparrow_k^V) \xrightarrow{\Phi(t_{M\uparrow_k^V})} \Phi(M \uparrow_k^V) \xrightarrow{\lambda} M \xrightarrow{0} \Phi(M \uparrow_k^V)[1].$$

We define $\mu := \Phi(f) \circ \Phi(t_{M\uparrow_k^V})$ and apply the octahedral axiom to this factorisation. Since Φ is assumed to be triangulated we get that $\Phi(\mathrm{cone}(f)) = \mathrm{cone}(\Phi(f))$ and we therefore obtain the commutative octahedral axiom diagram

$$
(I)\qquad
\begin{tikzcd}
\Phi(M\uparrow_k^V) \arrow[dr, "\Phi(t_{M\uparrow_k^V})"'] \arrow[drrr, "\mu"] & & & & M[1] & & \\
& \Phi(M\uparrow_k^V) \arrow[rr, "\Phi(f)"'] \arrow[ddr, "\lambda"'] & & \Phi(Q) \arrow[r, "\Phi(\varphi)"'] \arrow[dr, "\tau"] & \Phi(\mathrm{cone}(f)) \arrow[u] \arrow[rr, "0"'] & & \Phi(M\uparrow_k^V)[1] \\
& & & & \mathrm{cone}(\mu) \arrow[u, "\psi"'] \arrow[dr] & & \\
& & M \arrow[urr] \arrow[d, "0"] & & & \Phi(M\uparrow_k^V)[1] & \\
& & \Phi(M\uparrow_k^V)[1] & & & &
\end{tikzcd}
$$

where

$$M \longrightarrow \mathrm{cone}(\mu) \longrightarrow \Phi(\mathrm{cone}(f)) \longrightarrow M[1]$$

is a distinguished triangle. Since $\Phi(f)$ is a split monomorphism we further get that $\Phi(\varphi)$ is a split epimorphism with right inverse denoted $\omega : \Phi(\mathrm{cone}(f)) \longrightarrow \Phi(Q)$, i.e. $\Phi(\varphi) \circ \omega = \mathrm{id}_{\Phi(\mathrm{cone}(f))}$. But then

$$\psi \circ \tau \circ \omega = \Phi(\varphi) \circ \omega = \mathrm{id}_{\Phi(\mathrm{cone}(f))}$$

and therefore ψ is a split epimorphism. This shows that

$$\mathrm{cone}(\mu) \simeq \Phi(\mathrm{cone}(f)) \oplus M.$$

Hence

$$\Phi(\mathrm{cone}(f)) \oplus M \simeq \mathrm{cone}(\mu) \simeq \mathrm{cone}(\Phi(f \circ t_{M\uparrow_k^V})).$$

Observe that, since t is a natural transformation, we get

$$t_Q \circ f = f \circ t_{M\uparrow_k^V}.$$

The definition of categorical degeneration implies that we have a distinguished triangle

$$\Phi(Q) \stackrel{\Phi(t_Q)}{\longrightarrow} \Phi(Q) \longrightarrow N \longrightarrow \Phi(Q)[1].$$

Now we consider the factorisation

$$\mu = \Phi(t_Q \circ f) = \Phi(t_Q) \circ \Phi(f)$$

and obtain by the octahedral axiom the following diagram.

$$
(II)\qquad
\begin{tikzcd}
\Phi(M\uparrow_k^V) \arrow[rr, "\Phi(f)"] \arrow[drr, "\mu"'] & & \Phi(Q) \arrow[rr, "\Phi(\varphi)"] \arrow[d, "\Phi(t_Q)"] & & \Phi(\mathrm{cone}(f)) \arrow[r, "0"] \arrow[ddl, "\zeta"'] & \Phi(M\uparrow_k^V)[1] \\
& & \Phi(Q) \arrow[dr, "\tau"] \arrow[dd] & & & \\
& & & \Phi(\mathrm{cone}(t_Q \circ f)) \arrow[dl] \arrow[dr, "\nu"'] & & \\
& \Phi(\mathrm{cone}(f))[1] & N \arrow[l] \arrow[d] & & \Phi(M\uparrow_k^V)[1] & \\
& & \Phi(Q)[1] & & &
\end{tikzcd}
$$

where we thus obtain a distinguished triangle

$$\Phi(\mathrm{cone}(f)) \xrightarrow{\zeta} \Phi(\mathrm{cone}(t_Q \circ f)) \longrightarrow N \longrightarrow \Phi(\mathrm{cone}(f))[1].$$

Since $\Phi(\varphi)$ is a split epimorphism, Remark 6 shows that there is a unique choice $\zeta := \tau \circ \Phi(t_Q) \circ \omega$ so that the appearing diagrams are all commutative.

However, we already identified

$$\Phi(\mathrm{cone}(t_Q \circ f)) = \Phi(\mathrm{cone}(f \circ t_{M\uparrow_k^V})) \simeq \Phi(\mathrm{cone}(f)) \oplus M$$

so that we may put $Z := \Phi(\mathrm{cone}(f))$ to obtain a distinguished triangle in $\mathcal{C}_k$

$$Z \xrightarrow{\begin{pmatrix} v \\ u \end{pmatrix}} Z \oplus M \xrightarrow{\begin{pmatrix} \ell & m \end{pmatrix}} N \longrightarrow Z[1].$$

We need to show that v is nilpotent.

Recall that $Z = \Phi(\mathrm{cone}(f)) = \mathrm{cone}(\Phi(f))$. The octahedral axiom diagram (I) implied the fact that

$$\Phi(\mathrm{cone}(t_Q \circ f)) \simeq M \oplus \Phi(\mathrm{cone}(f))$$

and the octahedral axiom diagram (II) gives the distinguished triangle

$$\Phi(\mathrm{cone}(f)) \xrightarrow{\zeta} \Phi(\mathrm{cone}(t_Q \circ f)) \longrightarrow N \longrightarrow \Phi(\mathrm{cone}(f))[1].$$

Since by the octahedral diagram (I) we get a split distinguished triangle

$$M \longrightarrow \Phi(\mathrm{cone}(t_Q \circ f)) \xrightarrow{\psi} Z \longrightarrow M[1]$$

the canonical projection $\Phi(\mathrm{cone}(t_Q \circ f)) = Z \oplus M \longrightarrow Z$ is obtained by ψ, and hence

$$\begin{aligned} v &= \psi \circ \zeta \\ &= \psi \circ \tau \circ \Phi(t_Q) \circ \omega \\ &= \Phi(\varphi) \circ \Phi(t_Q) \circ \omega \\ &= \Phi(\varphi \circ t_Q) \circ \omega \\ &= \Phi(t_{\mathrm{cone}(f)} \circ \varphi) \circ \omega \\ &= \Phi(t_{\mathrm{cone}(f)}) \circ \Phi(\varphi) \circ \omega \\ &= \Phi(t_{\mathrm{cone}(f)}) \end{aligned}$$

where we used that ω is right inverse to $\Phi(\varphi)$, and that t is a natural transformation. But since $p(f)$ is an isomorphism in $\mathcal{C}_V^\circ[t^{-1}]$, we have that

$$p(\mathrm{cone}(f)) \simeq \mathrm{cone}(p(f)) \simeq 0$$

and we get that $t_{\mathrm{cone}(f)}$ is nilpotent by Lemma 7. Therefore $v = \Phi(t_{\mathrm{cone}(f)})$ is nilpotent.

We need to show that Z is in $\mathcal{C}_k^\circ$. We shall apply here Lemma 5 to the case

$$\begin{array}{ccc} \Phi(M\uparrow_k^V) & \stackrel{\Phi(f)}{\to} & \Phi(Q) \\ \downarrow \Phi(t_{M\uparrow_k^V}) & & \downarrow \Phi(t_Q) \\ \Phi(M\uparrow_k^V) & \stackrel{\Phi(f)}{\to} & \Phi(Q) \end{array}$$

Recall that in Lemma 5 we apply the octahedral axiom to the composition

$$\Phi(t_Q) \circ \Phi(f) = \mu = \Phi(f) \circ \Phi(t_{M\uparrow_k^V}).$$

The decomposition $\Phi(t_Q) \circ \Phi(f) = \mu$ gives the octahedral diagram (II) and the decomposition $\mu = \Phi(f) \circ \Phi(t_{M\uparrow_k^V})$ gives the octahedral diagram (I). The distinguished triangles

we obtain from these diagrams were determined in the first two steps of the proof. For the diagram (II) we get the distinguished triangle

$$Z \xrightarrow{\begin{pmatrix} v \\ u \end{pmatrix}} Z \oplus M \longrightarrow N \longrightarrow Z[1]$$

and for the diagram (I) we get the split distinguished triangle

$$M \longrightarrow Z \oplus M \xrightarrow{\begin{pmatrix} \mathrm{id}_Z & 0 \end{pmatrix}} Z \longrightarrow M[1].$$

We see that the map ε in

$$\begin{array}{ccccccc}
\Phi(M\uparrow_k^V) & \stackrel{\Phi(f)}{\to} & \Phi(Q) & \to & Z & \to & \Phi(M\uparrow_k^V)[1] \\
\downarrow \Phi(t_{M\uparrow_k^V}) & & \downarrow \Phi(t_Q) & & \downarrow \varepsilon & & \Phi(t_{M\uparrow_k^V})[1]\downarrow \\
\Phi(M\uparrow_k^V) & \stackrel{\Phi(f)}{\to} & \Phi(Q) & \to & Z & \to & \Phi(M\uparrow_k^V)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
M & \to & N & \to & C(\epsilon) & \to & M[1] \\
\downarrow & & \downarrow & & \downarrow & \text{“}(-)\text{”} & \downarrow \\
\Phi(M\uparrow_k^V)[1] & \stackrel{\Phi(f)[1]}{\to} & \Phi(Q)[1] & \to & Z[1] & \to & \Phi(M\uparrow_k^V)[2],
\end{array}$$

such that all rows and columns are distinguished triangles, is obtained as

$$\varepsilon = \begin{pmatrix} \mathrm{id}_Z & 0 \end{pmatrix} \cdot \begin{pmatrix} v \\ u \end{pmatrix} = v = \Phi(t_{\mathrm{cone}(f)}).$$

We have already seen that $t_{\mathrm{cone}(f)}$ is nilpotent, and so is therefore v. We remark that the left most column is a split triangle, and hence there is a unique choice for the map $M \longrightarrow N$, as can be obtained similarly as in Remark 6. We now use the second lower row distinguished triangle

$$M \to N \to \mathrm{cone}(\Phi(t_{\mathrm{cone}(f)})) \to M[1].$$

Since M and N are in the triangulated category $\mathcal{C}_k^\circ$, and since

$$M \to N \to \mathrm{cone}(\Phi(t_{\mathrm{cone}(f)})) \to M[1]$$

is a distinguished triangle, also $\mathrm{cone}(\Phi(t_{\mathrm{cone}(f)}))$ is in $\mathcal{C}_k^\circ$. We apply the octahedral axiom to the factorisation

$$\Phi(t^2_{\mathrm{cone}(f)}) = \Phi(t_{\mathrm{cone}(f)}) \circ \Phi(t_{\mathrm{cone}(f)})$$

and obtain a distinguished triangle

$$\mathrm{cone}(\Phi(t_{\mathrm{cone}(f)})) \to \mathrm{cone}(\Phi(t_{\mathrm{cone}(f)})^2) \to \mathrm{cone}(\Phi(t_{\mathrm{cone}(f)})) \to \mathrm{cone}(\Phi(t_{\mathrm{cone}(f)}))[1]$$

and hence also $\mathrm{cone}(\Phi(t_{\mathrm{cone}(f)})^2)$ is in $\mathcal{C}_k^\circ$. The octahedral axiom applied to

$$\Phi(t^n_{\mathrm{cone}(f)}) = \Phi(t^{n-1}_{\mathrm{cone}(f)}) \circ \Phi(t_{\mathrm{cone}(f)})$$

yields distinguished triangles

$$\mathrm{cone}(\Phi(t_{\mathrm{cone}(f)})) \to \mathrm{cone}(\Phi(t_{\mathrm{cone}(f)})^n) \to \mathrm{cone}(\Phi(t_{\mathrm{cone}(f)})^{n-1}) \to \mathrm{cone}(\Phi(t_{\mathrm{cone}(f)}))[1]$$

and by induction we get $\mathrm{cone}(\Phi(t_{\mathrm{cone}(f)})^n)$ is in $\mathcal{C}_k^\circ$ for all $n \geq 1$. But $\Phi(t_{\mathrm{cone}(f)})$ is nilpotent, and so the distinguished triangle

$$\Phi(\mathrm{cone}(f)) \xrightarrow{\Phi(t_{\mathrm{cone}(f)})^n} \Phi(\mathrm{cone}(f)) \to \mathrm{cone}(\Phi(t_{\mathrm{cone}(f)})^n) \to \Phi(\mathrm{cone}(f))[1]$$

shows that for large enough n we get $\Phi(t_{\mathrm{cone}(f)})^n = 0$ and so, $\Phi(\mathrm{cone}(f))$ is a direct summand of $\mathrm{cone}(\Phi(t_{\mathrm{cone}(f)})^n)$. But $\mathrm{cone}(\Phi(t_{\mathrm{cone}(f)})^n)$ is in $\mathcal{C}_k^\circ$ and idempotents split in $\mathcal{C}_k^\circ$. It follows that Z is in $\mathcal{C}_k^\circ$. $\square$

3.2. **Triangle degeneration implies categorical degeneration.** Following Keller [8] a triangulated k-category $\mathcal{C}$ is algebraic if $\mathcal{C}$ is the stable category of a Frobenius category.

Proposition 9. *Let $\mathcal{C}_k^0$ be the category of compact objects of an algebraic compactly generated triangulated k-category. If there is a distinguished triangle*

$$Z \xrightarrow{\begin{pmatrix} v \\ u \end{pmatrix}} Z \oplus M \xrightarrow{\begin{pmatrix} h & j \end{pmatrix}} N \longrightarrow Z[1]$$

in $\mathcal{C}_k^\circ$, with a nilpotent endomorphism v of Z, then $M \leq_{cdeg} N$ (in the sense of Definition 4).

Proof. Let $\mathcal{C}_k$ be a compactly generated algebraic triangulated k-category such that $\mathcal{C}_k^\circ$ is the full subcategory of its compact objects. By Keller [7, 4.3 Theorem], we know that $\mathcal{C}_k$ is triangle-equivalent to the derived category $\mathcal{D}(\mathcal{A})$ of a small dg k-category $\mathcal{A}$. We identify $\mathcal{C}_k = \mathcal{D}(\mathcal{A})$ throughout the rest of the proof.

Replacing each of the complexes Z, M and N by a homotopically projective resolution in the homotopy category $\mathcal{H}(\mathcal{A})$ and adding suitable contractible complexes if necessary, we can assume that the given triangle comes from a conflation

$$0 \longrightarrow Z \xrightarrow{\begin{pmatrix} v \\ u \end{pmatrix}} Z \oplus M \xrightarrow{\begin{pmatrix} h & j \end{pmatrix}} N \longrightarrow 0$$

in the complex category $\mathcal{C}(\mathcal{A})$, where in addition Z, M and N are finitely generated projective as graded $\mathcal{A}$-modules, when we forget the differential.

Let us put $V := k[[T]]$ and let $\mathcal{A}[[T]]$ be the dg V-category with the same objects as $\mathcal{A}$, where

$$\mathrm{Hom}_{\mathcal{A}[[T]]}(A, A') = \mathrm{Hom}_{\mathcal{A}}(A, A')[[T]] = \{\sum_{i=0}^{\infty} \alpha_i T^i \ : \ \alpha_i \in \mathrm{Hom}_{\mathcal{A}}(A, A'), \text{ for all } i \geq 0\},$$

for all $A, A' \in \mathcal{A}$, and the composition of morphisms is the obvious one. We then denote by $\mathcal{C}(\mathcal{A}[[T]])$, $\mathcal{H}(\mathcal{A}[[T]])$ and $\mathcal{C}_V := \mathcal{D}(\mathcal{A}[[T]])$ their associated categories of complexes, homotopy and derived category, respectively. We have an obvious functor

$$\begin{array}{rcl} ?\hat{\otimes} V : \mathcal{C}(\mathcal{A}) & \longrightarrow & \mathcal{C}(\mathcal{A}[[T]]) \\ M & \rightsquigarrow & M[[T]] \end{array}$$

where $M[[T]] : (\mathcal{A}[[T]])^{op} \longrightarrow \mathcal{C}(V)$ acts as $(M[[T]])(A) = M(A)[[T]]$ on objects and as $M[[T]](\sum_{i=0}^{\infty} \alpha_i T^i) = \sum_{i=0}^{\infty} M(\alpha_i) T^i$ on morphisms. The functor $?\hat{\otimes} V$ clearly takes conflations to conflations and null-homotopic maps to null-homotopic maps. We then get an induced triangulated functor

$$?\hat{\otimes} V : \mathcal{H}(\mathcal{A}) \longrightarrow \mathcal{H}(\mathcal{A}[[T]]).$$

This latter functor clearly takes acyclic complexes to acyclic complexes, so that we get a triangulated functor

$$\uparrow_k^V : \mathcal{C}_k = \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{A}[[T]]) =: \mathcal{C}_V.$$

If N is any object of $\mathcal{C}_V$, then $N(A)$ is a graded V-module, for each object A of $\mathcal{A}$. Multiplication by T gives then a morphism $t_N(A) : N(A) \longrightarrow N(A)$ of graded V-modules of degree zero, for each object A of $\mathcal{A}$. They define a morphism $t_N : N \longrightarrow N$ in $\mathcal{C}_V$. Moreover all the t_N define a natural transformation $t : \mathrm{id}_{\mathcal{C}_V} \longrightarrow \mathrm{id}_{\mathcal{C}_V}$ of triangulated functors. We claim that the data $(\mathcal{C}_k, \mathcal{C}_V, \mathcal{C}_V^\circ, \uparrow_k^V, t)$ give a degeneration data for $\mathcal{C}_k^\circ = \mathcal{D}^c(\mathcal{A})$ satisfying the requirements, where $\mathcal{C}_V^\circ = \mathcal{D}^c(\mathcal{A}[[T]])$ is the category of compact objects of $\mathcal{C}_V$. This will prove that $M \leq_{cdeg} N$.

There is an obvious 'restriction of scalars' triangulated functor $\phi : \mathcal{D}(\mathcal{A}[[T]]) \longrightarrow \mathcal{D}(\mathcal{A})$. Moreover, if X is an object of $\mathcal{C}_k^\circ = \mathcal{D}^c(\mathcal{A})$, then X is a direct summand of a finite iterated

extension of shifts of representable $\mathcal{A}$-modules $A^\wedge = \mathrm{Hom}_{\mathcal{A}}(?, A)$ (see [7, Theorem 5.3]). But we have an isomorphism in

$$(A^\wedge) \uparrow_k^V \cong \mathrm{Hom}_{\mathcal{A}[[T]]}(?, A),$$

for each object A of $\mathcal{A}$ (equivalently, of $\mathcal{A}[[T]]$). That is, the functor $\uparrow_k^V$ takes representable $\mathcal{A}$-modules to representable $\mathcal{A}[[T]]$-modules, and hence $(\mathcal{C}_k^\circ) \uparrow_k^V \subseteq \mathcal{C}_V^\circ$.

Let M be any object of $\mathcal{C}_k$. The morphism

$$\phi(t_{M\uparrow_k^V}) : \phi(M \uparrow_k^V) \longrightarrow \phi(M \uparrow_k^V),$$

when evaluated at an object A of $\mathcal{A}$ (or $\mathcal{A}[[T]]$), gives the morphism of k-modules

$$M(A)[[T]] \longrightarrow M(A)[[T]]$$

given by multiplication by T. We get a retraction

$$\pi_M(A) : M(A)[[T]] \longrightarrow M(A)[[T]]$$

for this map which takes $\sum_{i=0}^\infty m_i T^i$ to $\sum_{i=1}^\infty m_i T^{i-1}$. It is routine to see that this gives a retraction $\pi_M : \phi(M \uparrow_k^V) \longrightarrow \phi(M \uparrow_k^V)$ for $\phi(t_{M\uparrow_k^V})$. Moreover, in the abelian category $\mathcal{C}(\mathcal{A}[[T]])$, we have an exact sequence

$$0 \to M[[T]] \xrightarrow{\cdot T} M[[T]] \longrightarrow M[[T]]/TM[[T]] \to 0.$$

We then get a distinguished triangle

$$M \uparrow_k^V \xrightarrow{t_{M\uparrow_k^V}} M \uparrow_k^V \longrightarrow M[[T]]/TM[[T]] \longrightarrow M \uparrow_k^V [1]$$

in $\mathcal{D}(\mathcal{A}[[T]])$. It is clear that

$$\mathrm{cone}(\phi(t_{M\uparrow_k^V})) \cong \phi(M[[T]]/TM[[T]])$$

is isomorphic to M in $\mathcal{D}(\mathcal{A})$. Then all conditions of Definition 3 are satisfied.

We now check that $M \leq_{cdeg} N$ with respect to the above-defined degeneration data. We first claim that $\begin{pmatrix} v+T \\ u \end{pmatrix} : Z[[T]] \longrightarrow Z[[T]] \oplus M[[T]]$ is an inflation (=admissible monomorphism) in the exact category $\mathcal{C}(\mathcal{A}[[T]])$. For this we need to get a retraction for this map as in the category of graded $\mathcal{A}[[T]]$-modules. We look for a retraction of the form $\left(\left(\sum_{i=0}^\infty h_i T^i\right) \quad \left(\sum_{i=0}^\infty j_i T^i\right) \right)$, where $h_i : Z \longrightarrow Z$ and $j_i : M \longrightarrow Z$ are morphisms of graded $\mathcal{A}$-modules, for all $i \geq 0$. We already know that $\begin{pmatrix} v \\ u \end{pmatrix}$ allows such a retraction $\begin{pmatrix} h & j \end{pmatrix}$ in the category of graded $\mathcal{A}$-modules. The equation

$$\left(\left(\sum_{i=0}^\infty h_i T^i\right) \quad \left(\sum_{i=0}^\infty j_i T^i\right) \right) \cdot \begin{pmatrix} v+T \\ u \end{pmatrix} = \mathrm{id}_{Z[[T]]}$$

translates into

$$\begin{array}{rcl} j_0 u + h_0 v & = & \mathrm{id}_Z \\ h_0 + j_1 u + h_1 v & = & 0 \\ h_1 + j_2 u + h_2 v & = & 0 \\ \dots & \dots & \dots \\ h_{i-1} + j_i u + h_i v & = & 0 \\ \dots & \dots & \dots \end{array}$$

and so $h_0 = h$ and $j_0 = j$ solves the first equation. Since

$$\mathrm{id}_Z = \begin{pmatrix} h & j \end{pmatrix} \cdot \begin{pmatrix} v \\ u \end{pmatrix},$$

we get

$$-h = \begin{pmatrix} -h^2 & -hj \end{pmatrix} \cdot \begin{pmatrix} v \\ u \end{pmatrix}$$

and if we know h_{i-1}, then

$$-h_{i-1} = \begin{pmatrix} -h_{i-1}h & -h_{i-1}j \end{pmatrix} \cdot \begin{pmatrix} v \\ u \end{pmatrix},$$

so that we may put $h_i := -h_{i-1}h$ and $j_i := -h_{i-1}j$. This gives $h_i = (-1)^i h^{i+1}$ and $j_i = (-1)^i h^i j$ by an obvious induction. We then obtain a conflation in $\mathcal{C}(\mathcal{A}[[T]])$

$$(\dagger) \quad 0 \longrightarrow Z[[T]] \xrightarrow{\begin{pmatrix} v+T \\ u \end{pmatrix}} Z[[T]] \oplus M[[T]] \longrightarrow Q \longrightarrow 0$$

and we get a commutative diagram

$$\begin{array}{ccccccc}
0 \to & Z[[T]] & \xrightarrow{\begin{pmatrix} v+T \\ u \end{pmatrix}} & Z[[T]] \oplus M[[T]] & \to & Q & \to 0 \\
 & \downarrow T & & \downarrow \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} & & \downarrow T & \\
0 \to & Z[[T]] & \xrightarrow{\begin{pmatrix} v+T \\ u \end{pmatrix}} & Z[[T]] \oplus M[[T]] & \to & Q & \to 0
\end{array}$$

The canonical functor $p : \mathcal{C}_V^\circ \longrightarrow \mathcal{C}_V^\circ[t^{-1}]$ maps the endomorphism $v + T$ of $Z[[T]]$ onto an invertible endomorphism of $p(Z[[T]])$ since v is nilpotent and multiplication by T is an isomorphism in $\mathcal{C}_V^\circ[t^{-1}]$. The triangle given by the conflation $(\dagger)$ above is then mapped by p onto a split triangle in $\mathcal{C}_V^\circ[t^{-1}]$ such that the first component of the morphism

$$p(Z[[T]]) \xrightarrow{\begin{pmatrix} p(v+T) \\ p(u) \end{pmatrix}} p(Z[[T]]) \oplus p(Z[[T]])$$

is an isomorphism. We then get that $p(M \uparrow_k^V) = p(M[[T]])$ is isomorphic to $p(Q)$ in $\mathcal{C}_V^\circ[t^{-1}]$.

Moreover, multiplication by T gives a monomorphism $X[[T]] \to X[[T]]$ in the (abelian) category of complexes $\mathcal{C}(\mathcal{A}[[T]])$ of $\mathcal{A}[[T]]$-modules, for each $X \in \mathcal{C}(\mathcal{A})$. In addition, the forgetful functor $\mathcal{C}(\mathcal{A}[[T]]) \longrightarrow \mathrm{Gr} - \mathcal{A}[[T]]$ is exact, where $\mathrm{Gr} - \mathcal{A}[[T]]$ denotes the category of graded $\mathcal{A}[[T]]$-modules. This together with the split monomorphic condition of $v + T$ in $\mathrm{Gr} - \mathcal{A}[[T]]$ imply that the map $Q \xrightarrow{T} Q$ in the above diagram is also a monomorphism in $\mathcal{C}(\mathcal{A}[[T]])$. Applying now the kernel-cokernel lemma to that diagram, we get a short exact sequence

$$0 \longrightarrow Z[[T]]/TZ[[T]] \longrightarrow Z[[T]]/TZ[[T]] \oplus M[[T]]/TM[[T]] \longrightarrow Q/TQ \longrightarrow 0$$

in $\mathcal{C}(\mathcal{A}[[T]])$. Identifying $X[[T]]/TX[[T]]$ with X, for each object X of $\mathcal{C}(\mathcal{A})$, the first map is again $\begin{pmatrix} v \\ u \end{pmatrix}$, and hence $Q/TQ \simeq N$ in $\mathcal{C}(\mathcal{A})$. Hence $\Phi(\mathrm{cone}(t_Q)) \simeq N$ in $\mathcal{D}^c(\mathcal{A}) = \mathcal{C}_k^\circ$. This shows $M \leq_{cdeg} N$. $\square$

3.3. **The main result.** Our main result is the following.

Theorem 1. *Let k be a commutative ring and let $\mathcal{C}_k^\circ$ be a triangulated k-category with split idempotents. If M and N are objects of $\mathcal{C}_k^\circ$ and M degenerates to N in the categorical sense, then there is a distinguished triangle*

$$Z \xrightarrow{\begin{pmatrix} v \\ u \end{pmatrix}} Z \oplus M \longrightarrow N \longrightarrow Z[1]$$

with nilpotent endomorphism v. When $\mathcal{C}_k^\circ$ is equivalent to the category of compact objects of a compactly generated algebraic triangulated k-category, the converse is also true.

Proof. This is an immediate consequence of Proposition 8 and Proposition 9. $\square$

Remark 10. For any abelian category $\mathcal{A}$ define $M \leq_{ses} N$ if there is a short exact sequence $0 \to Z \xrightarrow{\begin{pmatrix} v \\ u \end{pmatrix}} Z \oplus M \to N \to 0$ without any further hypothesis on v. If $\mathcal{A}$ admits countable direct sums we showed in [6, Example 2.2.(1)], then $M \leq_{ses} N$ and $N \leq_{ses} M$ for each pair of objects N and M of $\mathcal{A}$. The example transposes immediately to the triangulated situation. We insist in the fact that there the endomorphism v is not nilpotent, as we supposed in the hypotheses of Theorem 1. If the endomorphism algebras of each of the concerned objects is artinian, then the condition on v to be nilpotent is superfluous in Theorem 1.

Remark 11. The definition that an object M degenerates to an object N of a triangulated category $\mathcal{D}_k^\circ$ in the categorical sense depends a priori on the category $\mathcal{D}_k^\circ$. Do we get the same concept if $\mathcal{D}_k^\circ$ is a full subcategory of a triangulated category $\mathcal{C}_k^\circ$? This question was answered at least partially in [18, Proposition 11].

Let k be a field and let $\mathcal{C}_k^\circ$ be the category of compact objects in an algebraic compactly generated triangulated k-category. If $\mathcal{D}_k^\circ$ is a full triangulated subcategory of $\mathcal{C}_k^\circ$, then for all objects M and N of $\mathcal{D}_k^\circ$ we get that $M \leq_{cdeg} N$ with respect to $\mathcal{D}_k^\circ$ if and only if $M \leq_{cdeg} N$ in $\mathcal{C}_k^\circ$.

4. Some concluding remarks on triangulated degeneration

Definition 12. Let $\mathcal{T}$ be any triangulated category. We shall denote by $\preceq_{cdeg}$ (resp. $\preceq_\Delta$, resp. $\preceq_{\Delta *}$) the smallest transitive relation in the class of objects of $\mathcal{T}$ satisfying that if $X \leq_{cdeg} Y$ (resp. there exists a distinguished triangle $Z \xrightarrow{\begin{pmatrix} v \\ u \end{pmatrix}} Z \oplus X \longrightarrow Y \longrightarrow Z[1]$, resp. there is such a triangle, with v a nilpotent endomorphism of Z), then $X \preceq_{cdeg} Y$ (resp. $X \preceq_\Delta Y$, resp. $X \preceq_{\Delta *} Y$).

Remark 13. We recall that Jensen, Su and the second author showed in [6] that if $\mathcal{T}$ is a triangulated category in which idempotent endomorphisms split, and in which the endomorphism ring of each object is artinian, then $\preceq_\Delta$ and $\preceq_{\Delta *}$ coincide and they can be defined in one step, i.e., $X \preceq_\Delta Y$ if, and only if, there is a distinguished triangle

$$Z \xrightarrow{\begin{pmatrix} v \\ u \end{pmatrix}} Z \oplus X \longrightarrow Y \longrightarrow Z[1].$$

If moreover $\mathcal{T}$ is a k-linear category over a commutative ring k such that the composition length (as k-module) of the homomorphism space between two objects is always finite, and if for each two objects X and Y in $\mathcal{T}$ there is $n \in \mathbb{Z}$ so that $\mathrm{Hom}_{\mathcal{T}}(X, Y[n]) = 0$, then $\preceq_\Delta$ is also antisymmetric. In this case, when $\mathcal{T}$ is skeletally small, the relation $\preceq_\Delta = \preceq_{\Delta *}$ is a partial order in the set of isomorphism classes of objects of $\mathcal{T}$.

Proposition 14. *Let k be a commutative ring. Let $\mathcal{T}$ be a triangulated category with split idempotents and suppose that for all objects X and Y the k-module $Hom_{\mathcal{T}}(X,Y)$ is of finite composition length. If $Hom_{\mathcal{T}}(M, M[1]) = 0$ then M is a minimal object with respect to the relation $\preceq_\Delta = \preceq_{\Delta *}$. More precisely, under these conditions $N \preceq_\Delta M$ implies $N \simeq M$.*

Proof. Note that for each object X of $\mathcal{C}$ we have that $\mathrm{End}_{\mathcal{T}}(X)$ is an Artinian k-algebra. Indeed $\mathrm{End}_{\mathcal{T}}(X)$ is a k-module of finite length and then $I := \mathrm{ann}_k(\mathrm{End}_{\mathcal{T}}(X))$ is an ideal of k such that k/I has finite length as k-module, and so k/I is Artinian. It follows that $\mathrm{End}_{\mathcal{T}}(X)$ is an Artinian k-algebra since it has finite length as a module over the artinian commutative ring k/I.

Let

$$Z \xrightarrow{\begin{pmatrix} u \\ v \end{pmatrix}} N \oplus Z \to M \to Z[1]$$

be a distinguished triangle, where we assume without loss of generality that v is nilpotent (see Remark 13). We need to show that $M \simeq N$. Apply $\mathrm{Hom}_{\mathcal{T}}(M, -) =: (M, -)$ to the distinguished triangle and part of the long exact sequence becomes

$$(M, Z) \to (M, N) \oplus (M, Z) \to (M, M) \to (M, Z[1]) \to (M, N[1]) \oplus (M, Z[1]) \to 0$$

using that $(M, M[1]) = 0$. This shows that $(M, N[1]) = 0$ and that the sequence

$$(M, Z) \longrightarrow (M, N) \oplus (M, Z) \longrightarrow (M, M) \to 0$$

is exact. Then the given distinguished triangle splits since id_M is in the image of the map $(M, N) \oplus (M, Z) \longrightarrow (M, M)$. This shows that $N \oplus Z \simeq M \oplus Z$. By [1, Theorem A.1] we obtain that $\mathcal{T}$ is a Krull-Schmidt category, and hence $N \cong M$. $\square$

Remark 15. If $\mathcal{T}$ is a triangulated category and if $Z \to Z \oplus M \to N \to Z[1]$ is a split distinguished triangle such that the induced endomorphism v on Z is in the Jacobson radical of the endomorphism ring of Z, then we can show that M is isomorphic to N, even if $\mathcal{T}$ is not a Krull-Schmidt category.

Remark 16. As observed by Yoshino [16, Remark 4.6] there cannot be a maximal element with respect to $\leq_\Delta$. Indeed,

$$X \longrightarrow 0 \longrightarrow X[1] \longrightarrow X[1]$$

is a distinguished triangle, and hence also for any object U of $\mathcal{T}$ we get that

$$X \xrightarrow{\begin{pmatrix} 0 \\ 0 \end{pmatrix}} U \oplus X \longrightarrow U \oplus X \oplus X[1] \longrightarrow X[1]$$

is a distinguished triangle. Choosing $Z := X$ this shows $U \leq_\Delta (U \oplus X \oplus X[1])$, i.e. that the object U degenerates to $U \oplus X \oplus X[1]$ for all objects X. Observe that here the endomorphism on X on the left of the distinguished triangle is actually 0, hence nilpotent, so that we are in the situation of Theorem 1.

For a commutative Noetherian ring R, we will denote by $R - fl$ the category of finite length R-modules.

Lemma 17. *Let R be a commutative ring and let $U \xrightarrow{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}} U \oplus V \xrightarrow{\begin{pmatrix} \gamma & \delta \end{pmatrix}} W \to 0$ be an exact sequence of finitely generated R-modules. Suppose that either one of the following conditions holds:*

(1) *R is Noetherian.*
(2) *The R-modules U, V and W have finite length.*

If length($Hom_R(V, Y)$) $=$ length($Hom_R(W, Y)$), for all R-modules of finite length Y, then $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is a monomorphism and the short exact sequence $0 \to U \xrightarrow{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}} U \oplus V \xrightarrow{\begin{pmatrix} \gamma & \delta \end{pmatrix}} W \to 0$ remains exact when we apply the functor $Hom_R(?, Y)$, for any R-module Y of finite length. Moreover, under condition (2) the sequence is split.

Proof. Let us fix an R-module Y of finite length. When we apply the contravariant functor $\mathrm{Hom}_R(?, Y)$, we get an exact sequence in $R - fl$

$$0 \to \mathrm{Hom}_R(W, Y) \xrightarrow{\begin{pmatrix} \gamma^* \\ \delta^* \end{pmatrix}} \mathrm{Hom}_R(U, Y) \oplus \mathrm{Hom}_R(V, Y) \xrightarrow{\begin{pmatrix} \alpha^* & \beta^* \end{pmatrix}} \mathrm{Hom}_R(U, Y).$$

By comparison of composition lengths, we get that

$$\text{length}(\text{Hom}_R(U,Y)) + \text{length}(\text{Hom}_R(V,Y)) \leq \text{length}(\text{Hom}_R(U,Y)) + \text{length}(\text{Hom}_R(W,Y))$$

and hence

$$\text{length}(\text{Hom}_R(V,Y)) \leq \text{length}(\text{Hom}_R(W,Y))$$

with equality if and only if $\begin{pmatrix}\alpha^* & \beta^*\end{pmatrix}$ is surjective.

Putting $f = \begin{pmatrix}\alpha \\ \beta\end{pmatrix}$, the proof reduces to check that if $f : M \longrightarrow N$ is a morphism of finitely generated R-modules such that $f^* : \text{Hom}_R(N,Y) \longrightarrow \text{Hom}_R(M,Y)$ is an epimorphism for all $Y \in R - fl$, then $\ker(f) = 0$.

Suppose that U, V and W are modules of finite length. Then we can take $Y = M$ and we get that id_M is in the image of f^*, which implies that f is a split monomorphism. This implies that $\ker(f) = 0$.

Suppose now that R is Noetherian. By localizing at maximal ideals, we can assume without loss of generality that R is a local ring. In such case, if $\mathbf{m}$ is the maximal ideal and we take $Y = M/\mathbf{m}^n M$, the fact that the projection $p : M \longrightarrow M/\mathbf{m}^n M$ is in the image of f^* implies that $\ker(f) \subseteq \mathbf{m}^n M$. It follows that $\ker(f) \subseteq \bigcap_{n\geq 0} \mathbf{m}^n M$, so that $\ker(f) = 0$ (see [10, Theorem 8.10]). □

Lemma 18. *Let k be a commutative ring, let $\mathcal{T}$ be a triangulated k-category, let $H : \mathcal{T} \longrightarrow k - mod$ be a cohomological functor, where $k - mod$ denotes the category of finitely generated k-modules. Put $H^j = H \circ (?[j])$, for each integer j, and suppose that either one of the following conditions holds:*

1. *k is Noetherian and, for each object X of $\mathcal{T}$, the k-module $H(X)$ is finitely generated and there is an integer $n = n_X$ such that $H^j(X) = 0$ for all $j > n$;*
2. *For each object X of $\mathcal{T}$, the k-module $H(X)$ is of finite length and one of the following subconditions holds:*
   - (a) *For each object X of $\mathcal{T}$, there is an integer $n = n_X$ such that $H^j(X) = 0$, for all $j > n$;*
   - (b) *For each object X of $\mathcal{T}$, there is an integer $n = n_X$ such that $H^j(X) = 0$, for all $j < n$*

If $Z \xrightarrow{\begin{pmatrix}v \\ u\end{pmatrix}} Z \oplus M \longrightarrow N \longrightarrow Z[1]$ is a distinguished triangle in $\mathcal{T}$ and $N \preceq_\Delta M$, then the long exact sequence associated to the cohomological functor H gives short exact sequences

$$0 \to H^j(Z) \longrightarrow H^j(Z) \oplus H^j(M) \longrightarrow H^j(N) \to 0,$$

for all integers j, which remain exact when we apply the functor $Hom_k(?,Y)$, for any k-module Y of finite length. In particular, they are all split exact under condition (2).

Proof. By definition of $\preceq_\Delta$, we have a sequence $M_0, M_1, \ldots, M_r$ of objects of $\mathcal{T}$, with $M_0 = M_r = M$ and $M_1 = N$, together with distinguished triangles

$$Z_i \longrightarrow Z_i \oplus M_i \longrightarrow M_{i+1} \longrightarrow Z_i[1]$$

$(i = 0, 1, \ldots, r-1)$.

We first assume that condition (1) or condition (2).(a) hold. We can take $n \in \mathbb{Z}$ such that $H^j(Z_i) = 0 = H^j(M_i)$, for all $i = 0, 1, \ldots, r-1$ and all $j > n$. We then get exact sequences of finitely generated k-modules

$$H^n(Z_i) \longrightarrow H^n(Z_i) \oplus H^n(M_i) \longrightarrow H^n(M_{i+1}) \to 0$$

$(i = 0, 1, \ldots, r-1)$. If Y is any k-module of finite length and we apply the functor $\text{Hom}_k(?,Y)$ then, counting composition lengths as in the proof of Lemma 17, we get that

$$\text{length}(\text{Hom}_k(H^n(M_i),Y)) \leq \text{length}(\text{Hom}_k(H^n(M_{i+1}),Y))$$

for $i = 0, 1, \ldots, r-1$. We then get that these inequalities are equalities since

$$\text{length}(\text{Hom}_k(H^n(M_0), Y)) = \text{length}(\text{Hom}_k(H^n(M_r), Y)).$$

Lemma 17 implies that we get short exact sequences

$$0 \to H^n(Z_i) \longrightarrow H^n(Z_i) \oplus H^n(M_i) \longrightarrow H^n(M_{i+1}) \to 0,$$

which remain exact when we apply $\text{Hom}_k(?, Y)$ for any k-module Y of finite length, and which split under hypothesis (2).(a).

The long exact sequence associated to the cohomological functor H then gives exact sequences

$$H^{n-1}(Z_i) \longrightarrow H^{n-1}(Z_i) \oplus H^{n-1}(M_i) \longrightarrow H^{n-1}(M_{i+1}) \to 0$$

$(i = 0, 1, \ldots, r-1)$. Repeating now iteratively for all $j < n$ the argument used for n, we obtain the result.

We now assume that condition (2).(b) holds and fix an integer n such that $H^j(Z_i) = 0 = H^j(M_i)$, for all $j < n$ and $i = 0, 1, \ldots, r-1$. We then get exact sequences

$$0 \to H^n(Z_i) \longrightarrow H^n(Z_i) \oplus H^n(M_i) \longrightarrow H^n(M_{i+1})$$

$(i = 0, 1, \ldots, r-1)$. It follows that $\text{length}(H^n(M_i)) \leq \text{length}(H^n(M_{i+1}))$, for each $i = 0, 1, \ldots, r-1$, and these inequalities are then equalities since $H^n(M_0) = H^n(M_r)$. We get short exact sequences

$$(*) \quad 0 \to H^n(Z_i) \longrightarrow H^n(Z_i) \oplus H^n(M_i) \longrightarrow H^n(M_{i+1}) \to 0.$$

When we take a k-module of finite length Y, apply the functor $\text{Hom}_k(?, Y)$ to these sequences and count composition lengths to obtain that

$$\text{length}(\text{Hom}_k(H^n(M_i), Y)) \leq \text{length}(\text{Hom}_k(H^n(M_{i+1}), Y)),$$

for each $i = 0, 1, \ldots, r-1$. Again, these inequalities are equalities and, using Lemma 17, we conclude that all the exact sequences $(*)$ split. Finally, using an argument dual to the one followed under conditions (1) or (2).(a), one easily shows by induction on $s \geq 0$ that the long exact sequence associated to H gives split short exact sequences

$$0 \to H^{n+s}(Z_i) \longrightarrow H^{n+s}(Z_i) \oplus H^{n+s}(M_i) \longrightarrow H^{n+s}(M_{i+1}) \to 0,$$

for $i = 0, 1, \ldots, r-1$. □

We are now ready to prove the main result of this section.

Theorem 2. *Let k be a commutative ring and let $\mathcal{T}$ be a skeletally small triangulated k-category with split idempotents. The following assertions hold:*

1. *Suppose that $Hom_{\mathcal{T}}(X, Y)$ is a k-module of finite length, for all objects X, Y of $\mathcal{T}$, and either one of the following two conditions hold for such objects:*
   (a) *There is an integer $n = n_{XY}$ such that $Hom_{\mathcal{T}}(X, Y[j]) = 0$, for all $j > n$;*
   (b) *There is an integer $n = n_{XY}$ such that $Hom_{\mathcal{T}}(X, Y[j]) = 0$, for all $j < n$.*
   Then $\preceq_\Delta = \preceq_{\Delta*}$ is a partial order in the set of isomorphism classes of objects of $\mathcal{T}$.
2. *Suppose that k is Noetherian, that $Hom_{\mathcal{T}}(X, Y)$ is a finitely generated k-module, for all objects X, Y of $\mathcal{T}$, and that there is an $n = n_{XY}$ such that $Hom_{\mathcal{T}}(X, Y[j]) = 0$ for all $j > n$. If $M \preceq_\Delta N$ and $N \preceq_\Delta M$ then there is an object Z of $\mathcal{T}$ such that $M \oplus Z$ and $N \oplus Z$ are isomorphic.*

Proof. For the equality $\preceq_\Delta = \preceq_{\Delta*}$ in Assertion (1), see Remark 13. Both in assertion (1) and (2), the relation $\preceq_\Delta$ is reflexive and transitive. Let us assume that $M \preceq_\Delta N$ and $N \preceq_\Delta M$. We then have a sequence of objects $M_0, M_1, \ldots, M_r$ in $\mathcal{T}$ such that $M_0 = M = M_r$ and $M_s = N$, for some $s = 0, 1, \ldots, r$ and we have distinguished triangles

$$(**) \qquad Z_i \longrightarrow Z_i \oplus M_i \longrightarrow M_{i+1} \longrightarrow Z_i[1]$$

$(i = 0, 1, \dots, r-1)$. Let X be any object of $\mathcal{T}$ and consider the cohomological functor $\mathrm{Hom}_{\mathcal{T}}(X, ?) : \mathcal{T} \longrightarrow k-\mathrm{mod}$. By Lemma 18, under any of the assumptions, we have exact sequences

$$0 \to \mathrm{Hom}_{\mathcal{T}}(X, Z_i[j]) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(X, Z_i[j]) \oplus \mathrm{Hom}_{\mathcal{T}}(X, M_i[j]) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(X, M_{i+1}[j]) \to 0,$$

for all $j \in \mathbb{Z}$ and $i = 0, 1, \dots, r-1$. Fixing i and putting $X = M_{i+1}$ and $j = 0$, we get that all the triangles $(**)$ split. Therefore $M_i \oplus Z_i \cong M_{i+1} \oplus Z_i$, for all $i = 0, 1, \dots, r-1$, which easily implies for the object $Z = Z_0 \oplus \cdots \oplus Z_{s-1}$ of $\mathcal{T}$ that $M \oplus Z \cong N \oplus Z$.

When $\mathrm{Hom}_{\mathcal{T}}(X, Y)$ is a k-module of finite length, for all objects X, Y of $\mathcal{T}$, we have that $\mathcal{T}$ is a Krull-Schmidt category. Indeed $\mathrm{End}_{\mathcal{T}}(X)$ is a k-module of finite length and then $I := \mathrm{ann}_k(\mathrm{End}_{\mathcal{T}}(X))$ is an ideal of k such that k/I is Artinian. It follows that $\mathrm{End}_{\mathcal{T}}(X)$ is an Artinian k-algebra, for each object X of $\mathcal{T}$. By [1, Theorem A.1] we get that $\mathcal{T}$ is a Krull-Schmidt category. Hence, from the isomorphism $M \oplus Z \cong N \oplus Z$ we get that $M \cong N$. □

Example 19. We recall Example 2.2.(2) of [6]. Let G be the generalised quaternion group of order 32. Swan gives an ideal $\mathfrak{a}$ of $\mathbb{Z}G$ such that $\mathbb{Z}G \oplus \mathfrak{a} \simeq \mathbb{Z}G \oplus \mathbb{Z}G$ as $\mathbb{Z}G$-modules, and such that $\mathfrak{a} \not\simeq \mathbb{Z}G$. This implies that $\mathbb{Z}G \leq_\Delta \mathfrak{a} \leq_\Delta \mathbb{Z}G$ when we choose $\mathcal{T} = D^b(\mathbb{Z}G)$.

Recall that a ring A is called a an *Artin algebra* (resp. a *Noether algebra*) when its center is Artinian (resp. Noetherian) and A is finitely generated as a module over that center.

Corollary 20. *If A is an Artin algebra and $\mathcal{T} = D^b(A)$, then the relations $\preceq_{cdeg}$, $\preceq_\Delta$ and $\preceq_{\Delta*}$ coincide in $D^b(A)$ and are partial orders in the set of isomorphism classes of objects of $D^b(A)$.*

Proof. By Remark 13, we know that $\preceq_\Delta = \preceq_{\Delta*}$. Let M, N be any objects of $D^b(A)$ and suppose that $M \preceq_{cdeg} N$. By definition of $\preceq_{cdeg}$, there is a sequence of objects $M_0, M_1, \dots, M_r$ in $D^b(A)$ such that $M_0 = M$ and $M_r = N$ and such that $M_i \leq_{cdeg} M_{i+1}$ for all $i = 0, 1, \dots, r-1$. By Theorem 1, we get that $M \preceq_\Delta N$. For the converse implication, note that $D^b(A)$ is equivalent to the subcategory of compact objects of a compactly generated algebraic triangulated category (see [9]). Then, again by Theorem 1, we get that $M \preceq_\Delta N$ implies $M \preceq_{cdeg} N$.

Finally, $\mathcal{T} = D^b(A)$ satisfies condition (1).(b) of Theorem 2 and, hence, $\preceq_\Delta = \preceq_{cdeg}$ is a partial order in the set of isomorphism classes of objects of $D^b(A)$. □

Recall that a dg algebra A is *homologically upper bounded* when there is an integer m such that $H^n(A) = 0$, for all $n > m$.

Corollary 21. *Let k be Noetherian commutative and let $A = \oplus_{n \in \mathbb{Z}} A^n$ be a dg k-algebra such that $H^n(A)$ is a finitely generated k-module, for all $n \in \mathbb{Z}$. Suppose that A is homologically upper bounded (e.g., a Noether algebra viewed as a dg algebra concentrated in degree 0) and let per(A) be the category of compact objects of $D(A)$. The relations $\preceq_{\Delta^*}$ and $\preceq_{cdeg}$ coincide in per(A). Moreover, if M, N are objects of per(A) such that $M \preceq_\Delta N$ and $N \preceq_\Delta M$ (in particular, if $M \preceq_{cdeg} N$ and $N \preceq_{cdeg} M$), then the following assertions hold:*

1. *There is $Z \in$ per(A) such that $Z \oplus M$ and $Z \oplus N$ are isomorphic in $D(A)$;*
2. *The $k_\wp$-modules $H^j(M)_\wp$ and $H^j(N)_\wp$ are isomorphic, for all $j \in \mathbb{Z}$ and all prime ideals $\wp$ of k.*

Proof. That $\preceq_{\Delta^*}$ and $\preceq_{cdeg}$ coincide in per(A) is a direct consequence of Theorem 1 and the definition of these relations. Assertion (1) will follow from Theorem 2 once we check that if X, Y are objects of per(A), then $\mathrm{Hom}_{D(A)}(X, Y)$ is a finitely generated k-module and $\mathrm{Hom}_{D(A)}(X, Y[j]) = 0$ for $j >> 0$. Indeed, each object of per(A) is a direct summand of a finite iterated extension of shifts $A[n]$, with $n \in \mathbb{Z}$ (see [7, Theorem 5.3]). This reduces the proof to the case when $X = A[m]$ and $Y = A[n]$, for some $m, n \in \mathbb{Z}$. We have that

$\mathrm{Hom}_{D(A)}(A[m], A[n]) \cong H^{n-m}(A)$, which is a finitely generated k-module. On the other hand, we have an equality

$$\begin{aligned}\mathrm{Hom}_{D(A)}(A[m], A[n][j]) &= \mathrm{Hom}_{D(A)}(A[m], A[n+j]) \\ &\cong \mathrm{Hom}_{D(A)}(A, A[n-m+j]) \\ &\cong H^{n-m+j}(A).\end{aligned}$$

This is zero for $j >> 0$ due to the homologically upper bounded condition of A.

Given an isomorphism $Z \oplus M \cong Z \oplus N$, we get that $\mathrm{length}(\mathrm{Hom}_k(H^j(M), Y)) = \mathrm{length}(\mathrm{Hom}_k(H^j(M), Y))$, for each k-module Y of finite length. Assertion (2) then follows from [14, Theorem 2.2]. □

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